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EVERY SIMPLE HIGHER DIMENSIONAL NONCOMMUTATIVE TORUS IS AN AT ALGEBRA

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Abstract. We prove that every simple higher dimensional noncommutative torus is an AT algebra.

0. Introduction

A higher dimensional noncommutative torus is the universal C*-algebra generated by unitaries which commute up to specified scalars. Thus, it is a generalization of the rotation algebra A_θ to more generators. The commutation relations are determined by a real skew symmetric matrix; see Notation 1.1 for a precise formulation. In this paper, we prove that every simple higher dimensional noncommutative torus is an AT algebra, that is, a direct limit of finite direct sums of C*-algebras of the form $C(S^1, M_n)$ for varying values of n.

The first result in this direction is the Elliott-Evans Theorem [10] for the ordinary irrational rotation algebras, which we use here as the initial step of an induction argument. Without giving a complete list of later work, we mention four highlights. All simple three dimensional noncommutative tori were shown to be AT algebras by Q. Lin [25]. In arbitrary dimension, "most" simple higher dimensional noncommutative tori were shown to be AT algebras by Boca [4]. Kishimoto (Corollary 6.6 of [18]) obtained this result in all cases in which, in the skew symmetric matrix giving the commutation relations, the entries above the diagonal are rationally independent, as well as some others. Theorem 3.14 of [22] shows that the crossed product of $(S^1)^d$ by a minimal rotation is an AT algebra; in this case, most of the entries of the relevant skew symmetric matrix are zero.

Our proof is by induction on the number of generators. Every higher dimensional noncommutative torus can be written as an iterated crossed product by $\mathbb{Z}$, and the proof of Kishimoto's result uses an inductive argument which works whenever the intermediate crossed products are all simple. One has some choice here: different choices of the commutation relations may well give the same C*-algebra. As a very simple example, one might simply write the generators in a different order. Unfortunately, it seems not to be possible in general to choose commutation relations to give the same algebra, or even a Morita equivalent algebra (see [35]), and in such a way that Kishimoto's method applies, or even in such a way as to get a tensor product of algebras to which this method applies. However, if one allows one more kind of modification, namely the replacements of unitary generators by

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integer powers of themselves, then it is always possible to replace a noncommutative torus by a tensor product of algebras covered by Kishimoto's method. The new algebra isn't isomorphic, or even Morita equivalent, to the original. But if one replaces only one generator, the new algebra is the fixed point algebra of a tracially approximately inner action [28] of a finite cyclic group which has the tracial Rokhlin property [28]. As proved in [28], this operation thus preserves tracial rank zero. Because K_0 and K_1 are torsion free, H. Lin's classification theorem for simple nuclear C*-algebras with tracial rank zero, Theorem 5.2 of [23], shows that this operation preserves the property of being an AT algebra. Indeed, the tracial Rokhlin property was introduced specifically for this purpose. It is not possible to substitute the Rokhlin property of [15] and [16] in this argument. As we show in Corollary 3.15, the relevant action does not have the Rokhlin property.

This paper is organized as follows. Section 1 contains various preliminaries. There is little that is really new, but our presentation gives the material in a convenient form and establishes notation for the rest of the paper. In Section 2, we prove that if A is a simple higher dimensional noncommutative torus, then the automorphism which multiplies one of the standard unitary generators by $\exp(2\pi i/n)$ generates an action of $\mathbb{Z}_n$ with the tracial Rokhlin property. In Section 3, we use this result, Kishimoto's result, and H. Lin's classification theorem [23], to construct an inductive proof that every simple noncommutative torus is an AT algebra, and we obtain several corollaries.

This paper contains the material of Sections 5 through 7 of the unpublished long preprint [27].

We use the notation $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$; the p-adic integers will not appear in this paper. If A is a C*-algebra and $\alpha \colon A \to A$ is an automorphism such that $\alpha^n = \mathrm{id}_A$, then we write $C^*(\mathbb{Z}_n, A, \alpha)$ for the crossed product of A by the action of $\mathbb{Z}_n$ generated by α. We write $p \precsim q$ to mean that the projection p is Murray-von Neumann equivalent to a subprojection of q, and $p \sim q$ to mean that p is Murray-von Neumann equivalent to q. Also, $[a, b]$ denotes the additive commutator $ab - ba$.

I am grateful to Marc Rieffel for discussions concerning higher dimensional noncommutative tori, and in particular for pointing out that it was not known whether a higher dimensional noncommutative torus is isomorphic to its opposite algebra. See Corollary 3.14; this remains open in the nonsimple case. I would like to thank Hiroyuki Osaka for carefully reading earlier versions of this paper, and catching a number of misprints and suggesting many improvements, and Hanfeng Li for useful comments, including the improvement of Lemma 2.11.

1. Higher dimensional noncommutative tori

In this section we present, in a form convenient for our purposes, some mostly standard facts about higher dimensional noncommutative tori and about irrational rotation algebras.

Notation 1.1. Let θ be a skew symmetric real $d \times d$ matrix. The *noncommutative torus* A_θ is by definition [34] the universal C*-algebra generated by unitaries $u_1, u_2, \ldots, u_d$ subject to the relations

$$u_k u_j = \exp(2\pi i \theta_{j,k}) u_j u_k$$

for $1 \leq j, k \leq d$. (Of course, if all $\theta_{j,k}$ are integers, it is not really noncommutative.)

Some authors use $\theta_{k,j}$ in the commutation relation instead. See for example [17].

Remark 1.2. We note (see the beginning of Section 4 of [32] and the introduction to [35]) that A_θ is the universal C*-algebra generated by unitaries u_x, for $x \in \mathbb{Z}^d$, subject to the relations

$$u_y u_x = \exp(\pi i \langle x, \theta(y) \rangle) u_{x+y}$$

for $x,\, y \in \mathbb{Z}^d$.

It follows that if $B \in \mathrm{GL}_d(\mathbb{Z})$, and if B^{t} denotes the transpose of B, then $A_{B^{\mathrm{t}} \theta B} \cong A_\theta$. That is, A_θ is unchanged if θ is rewritten in terms of some other basis of $\mathbb{Z}^d$.

Remark 1.3. Let α be a skew symmetric real bicharacter on $\mathbb{Z}^d$, that is, a $\mathbb{Z}$-bilinear function $\alpha \colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$ such that $\alpha(x, y) = -\alpha(y, x)$ for all $x,\, y \in \mathbb{Z}^d$. For any basis $(b_1, b_2, \ldots, b_d)$ of $\mathbb{Z}^d$, there is a unique skew symmetric real $d \times d$ matrix θ such that

$$\alpha\left(\sum\nolimits_{k=1}^{d} x_k b_k,\, \sum\nolimits_{k=1}^{d} y_k b_k \right) = \sum\nolimits_{j,k=1}^{d} x_j \theta_{j,k} y_k$$

for all $x,\, y \in \mathbb{Z}^d$. We define $A_\alpha = A_\theta$. Remark 1.2 shows that this C*-algebra is independent of the choice of basis.

Remark 1.4. Let α be a skew symmetric real bicharacter on $\mathbb{Z}^d$, and let $H \subset \mathbb{Z}^d$ be a subgroup. Then $H \cong \mathbb{Z}^m$ for some $m \leq d$. By abuse of notation, we write $\alpha|_H$ for the restriction of α to $H \times H \subset \mathbb{Z}^d \times \mathbb{Z}^d$. There is a noncommutative torus $A_{\alpha|_H}$ by Remark 1.3, which does not depend on the choice of the isomorphism $H \cong \mathbb{Z}^m$.

For a skew symmetric real $d \times d$ matrix θ and a subgroup $H \subset \mathbb{Z}^d$ with a specified ordered basis, we write $\theta|_H$ for the matrix in that basis of the restriction to H of the real bicharacter $(x, y) \mapsto \langle x, \theta y \rangle$. For subgroups such as $\mathbb{Z}^m \times \{0\}$ or $\mathbb{Z}^m \times \{0\} \times \mathbb{Z}^l$, we use without comment the obvious basis.

We formalize a remark made in 1.7 of [8], according to which every noncommutative torus can be obtained as a repeated crossed product by $\mathbb{Z}$.

Lemma 1.5. Let α be a skew symmetric real bicharacter on $\mathbb{Z}^d$. Then there is an automorphism φ of $A_{\alpha|_{\mathbb{Z}^{d-1} \times \{0\}}}$ which is homotopic to the identity and such that

$$A_\alpha \cong C^*(\mathbb{Z},\, A_{\alpha|_{\mathbb{Z}^{d-1} \times \{0\}}},\, \varphi).$$

Proof. Let θ be the matrix of α in the standard basis. Let $\beta = \alpha|_{\mathbb{Z}^{d-1} \times \{0\}}$. Then the matrix of β is $(\theta_{j,k})_{1 \leq j,k \leq d-1}$. Let $u_1,\, u_2,\, \ldots,\, u_{d-1}$ be the standard generators of A_β. Then φ is determined by $\varphi(u_j) = \exp(2\pi i \alpha_{j,d}) u_j$. It is clear that φ is homotopic to the identity. ∎

The following definition is essentially from Section 1.1 of [37].

Definition 1.6. The skew symmetric real $d \times d$ matrix θ is *nondegenerate* if whenever $x \in \mathbb{Z}^d$ satisfies $\exp(2\pi i \langle x, \theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$, then $x = 0$. Otherwise, θ is *degenerate*. We similarly refer to degeneracy and nondegeneracy of a skew symmetric real bicharacter on $\mathbb{Z}^d$.

Lemma 1.7. Let θ be a skew symmetric real $d \times d$ matrix. Then θ is degenerate if and only if there exists $x \in \mathbb{Q}^d \setminus \{0\}$ such that $\langle x,\, \theta y \rangle \in \mathbb{Q}$ for all $y \in \mathbb{Q}^d$.

Proof. If θ is degenerate, choose $w \neq 0$ such that $\exp(2\pi i \langle w, \theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$. Then $\langle w, \theta y \rangle \in \mathbb{Z}$ for all $y \in \mathbb{Z}^d$. If now $y \in \mathbb{Q}^d$ is arbitrary, then there exists $m \in \mathbb{Z} \setminus \{0\}$ such that $my \in \mathbb{Z}^d$. So

$$\langle w, \theta y \rangle = \tfrac{1}{m} \langle w, \theta(my) \rangle \in \tfrac{1}{m}\mathbb{Z} \subset \mathbb{Q}.$$

Conversely, assume $x \in \mathbb{Q}^d \setminus \{0\}$ and $\langle x, \, \theta y \rangle \in \mathbb{Q}$ for all $y \in \mathbb{Q}^d$. Choose $m \in \mathbb{Z}$ with $m > 0$ such that $m \langle x, \, \theta \delta_k \rangle \in \mathbb{Z}$ for $1 \leq k \leq d$. Then $mx \neq 0$ and $\exp(2\pi i \langle mx, \theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$. ∎

Lemma 1.8. Let θ be a skew symmetric real $d \times d$ matrix. Let $B \in \mathrm{GL}_d(\mathbb{Q})$. Then $B^{\mathrm{t}} \theta B$ is nondegenerate if and only if θ is nondegenerate.

Proof. It suffices to prove one direction. Suppose θ is degenerate. By Lemma 1.7, there is $x \in \mathbb{Q}^d \setminus \{0\}$ such that $\langle x, \, \theta y \rangle \in \mathbb{Q}$ for all $y \in \mathbb{Q}^d$. Then $B^{-1} x \in \mathbb{Q}^d \setminus \{0\}$ and

$$\langle B^{-1} x, \, B^{\mathrm{t}} \theta B y \rangle = \langle x, \, \theta B y \rangle \in \mathbb{Q}$$

for all $y \in \mathbb{Q}^d$. So $B^{\mathrm{t}} \theta B$ is degenerate. ∎

The following result is well known.

Theorem 1.9. The C*-algebra A_θ of Notation 1.1 is simple if and only if θ is nondegenerate. Moreover, if A_θ is simple it has a unique tracial state.

Proof. If θ is nondegenerate, then A_θ is simple by Theorem 3.7 of [37]. (Note the standing assumption of nondegeneracy throughout Section 3 of [37].)

When A_θ is simple, the proof of Lemma 3.1 of [37] shows that A_θ can have at most one tracial state. Existence of a tracial state is well known, or can be obtained from Lemma 1.5 by induction on n.

If θ is degenerate, then we follow 1.8 of [8]. Choose $n \in \mathbb{Z}^d \setminus \{0\}$ such that $\exp(2\pi i \langle n, \theta y \rangle) = 1$ for all $y \in \mathbb{Z}^d$. Then $v = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}$ is a nontrivial element of the center of A_θ, which is therefore not simple. ∎

Finally, for reference and to establish notation, we recall several facts about the ordinary rotation algebras (the case $d = 2$). (As far as we know, the last lemma has not appeared before, but its proof is easy.) We will consider various embeddings of rotation algebras into higher dimensional noncommutative tori. Therefore, for $\eta \in \mathbb{R}$ we let v_η and w_η denote the standard unitary generators of A_η, satisfying $w_\eta v_\eta = \exp(2\pi i \eta) v_\eta w_\eta$. In this way, we avoid confusion with the generators $u_1, u_2, \ldots, u_d$ of a higher dimensional noncommutative torus A_θ.

The proof of the next theorem is contained in Theorem 1.1 and Proposition 1.3 of [1]. Also see Corollary 3.6 and Definition 3.3 of [33]. We refer to [5] for information on continuous fields of C*-algebras. See especially Sections 10.1 and 10.3.

Theorem 1.10. For $\eta \in \mathbb{R}$ let A_η be the rotation algebra, with generators as described above. Let A be the C*-algebra of the discrete Heisenberg group, which is the universal C*-algebra generated by unitaries v, w, z subject to the relations

$$wv = zvw, \quad zv = vz, \quad \text{and} \quad zw = wz.$$

Then there is a continuous field of C*-algebras over S^1 whose fiber over $\exp(2\pi i \eta)$ is A_η, whose C*-algebra of continuous sections is A, and such that the evaluation map $\mathrm{ev}_\eta \colon A \to A_\eta$ of sections at $\exp(2\pi i \eta)$ is determined by

$$\mathrm{ev}_\eta(v) = v_\eta, \quad \mathrm{ev}_\eta(w) = w_\eta, \quad \text{and} \quad \mathrm{ev}_\eta(z) = \exp(2\pi i \eta) \cdot 1.$$

Since we will only formally deal with one continuous field in this paper, the following notation is unambiguous.

Notation 1.11. For a subset $E \subset S^1$, we let $\Gamma(E)$ be the set of continuous sections of the continuous field of Theorem 1.10 over E. (See 10.1.6 of [5].)

For any such section a, we further write $a(\eta)$ for $a(\exp(2\pi i \eta))$. No confusion should arise.

Lemma 1.12. Let the notation be as in Theorem 1.10 and Notation 1.11. Let τ_η be the standard trace on A_η, satisfying $\tau(1) = 1$ and $\tau_\eta(v_\eta^m w_\eta^n) = 0$ unless $m = n = 0$. (If A_η is viewed as a crossed product by rotation on the circle, then τ_η comes from normalized Haar measure on the circle.) Let $U \subset S^1$ be an open set, and let $a \in \Gamma(U)$. Then $\eta \mapsto \tau_\eta(a(\eta))$ is continuous.

Proof. We check continuity at η_0. Choose a continuous function $h \colon S^1 \to [0, 1]$ such that $\mathrm{supp}(h) \subset U$ and such that $h = 1$ on a neighborhood of η_0. Then it suffices to consider the continuous section ha in place of a. Now ha is the restriction to U of a continuous section b defined on all of S^1, satisfying $b(\zeta) = 0$ for $\zeta \notin U$. Accordingly, we may restrict to the case $U = S^1$. Then $a \in A$.

From the formulas

$$\mathrm{ev}_\eta(v) = v_\eta, \quad \mathrm{ev}_\eta(w) = w_\eta, \quad \text{and} \quad \mathrm{ev}_\eta(z) = \exp(2\pi i \eta) \cdot 1$$

and the definition of τ_η, it is immediate that if b is any (noncommutative) monomial in v, w, z, and their adjoints, then $\eta \mapsto \tau_\eta(b(\eta))$ is continuous. Therefore the same holds for any noncommutative polynomial, and hence for any norm limit of noncommutative polynomials, including a. ∎

Lemma 1.13. Let the notation be as in Theorem 1.10 and Lemma 1.12. Let $\eta \in \mathbb{R} \setminus \mathbb{Q}$. Let $n \in \mathbb{N}$, let $\omega = \exp(2\pi i/n)$, and let $\alpha \colon A_\eta \to A_\eta$ be the unique automorphism satisfying $\alpha(v_\eta) = \omega v_\eta$ and $\alpha(w_\eta) = w_\eta$. Then for every $\varepsilon > 0$ there exist mutually orthogonal projections $e_0, e_1, \ldots, e_{n-1}$ such that (with $e_n = e_0$) we have $\alpha(e_j) = e_{j+1}$ for $0 \leq j \leq n-1$, and such that $1 - n\tau_\eta(e_0) < \varepsilon$.

Proof. Set $\varepsilon_0 = \frac{1}{4n}\varepsilon$. Let $f \colon S^1 \to [0, 1]$ be a continuous function such that $\mathrm{supp}(f)$ is contained in the open arc from 1 to ω, and such that $f(\zeta) = 1$ for all ζ in the closed arc from $\exp(2\pi i \varepsilon_0)$ to $\exp\left(2\pi i \left[\frac{1}{n} - \varepsilon_0\right]\right)$. Then $f(v_\eta)$ is a positive element of A_η with $\|f(v_\eta)\| \leq 1$ and $\tau_\eta(f(v_\eta)) \geq \frac{1}{n} - 2\varepsilon_0$. Since A_η has real rank zero (see Remark 6 of [10], or Theorem 1.5 of [3]), there is a projection e_0 in the hereditary subalgebra B of A_η generated by $f(v_\eta)$ such that $\|e_0 f(v_\eta) - f(v_\eta)\| < \varepsilon_0$. Therefore $\|e_0 f(v_\eta) e_0 - f(v_\eta)\| < 2\varepsilon_0$. Since $e_0 f(v_\eta) e_0 \leq e_0$, it follows that

$$\tau_\eta(e_0) \geq \tau_\eta(e_0 f(v_\eta) e_0) > \tau_\eta(f(v_\eta)) - 2\varepsilon_0 \geq \tfrac{1}{n} - 4\varepsilon_0.$$

We have $\alpha^k(f(v_\eta))\alpha^l(f(v_\eta)) = 0$ for $0 \leq k, l \leq n-1$ and $k \neq l$. Therefore $\alpha^k(B)\alpha^l(B) = \{0\}$ for such k and l, whence also $\alpha^k(e_0)\alpha^l(e_0) = 0$. Define $e_k = \alpha^k(e_0)$ for $0 \leq k \leq n-1$. Then $e_0, e_1, \ldots, e_{n-1}$ are mutually orthogonal projections such that $\alpha(e_j) = e_{j+1}$ for $0 \leq j \leq n-1$. Moreover,

$$1 - n\tau_\eta(e_0) < 1 - n\left(\tfrac{1}{n} - 4\varepsilon_0\right) = 4n\varepsilon_0 = \varepsilon,$$

as desired. ∎

2. The tracial Rokhlin property and higher dimensional noncommutative tori

In this section, we prove that if θ is nondegenerate, then the action of $\mathbb{Z}_n$ which multiplies one of the standard generators of A_θ by a primitive n-th root of 1 has the tracial Rokhlin property. We note for comparison the related result in Section 6 of [17]: if $\alpha \in \mathrm{Aut}(A_\theta)$ is of the form $\alpha(u_j) = \lambda_j u_j$, with $\lambda_1, \lambda_2, \ldots, \lambda_n \in S^1$, and if all positive powers of α are outer, then α has the Rokhlin property. However, our action of $\mathbb{Z}_n$ does not have the Rokhlin property. See Corollary 3.15 below.

One might hope to prove the tracial Rokhlin property fairly directly, using one of the criteria in Section 5 of [6] or Section 1 of [29]. This approach does not work, because these criteria all assume that the C*-algebra involved is already known to have tracial rank zero, while our argument requires knowing that the action on a particular higher dimensional noncommutative torus has the tracial Rokhlin property before we know that it has tracial rank zero. Instead, as is done in the proof in [2] that A_θ has real rank zero, and analogously to Section 6 of [17], we will reduce to a construction in the ordinary irrational rotation algebras. The idea is to find an approximately central copy of an ordinary irrational rotation algebra A_η, such that the restriction to it of our action is the one in Lemma 1.13. Since the projections in A_η must be chosen ahead of time, at least approximately, we must require that η be arbitrarily close to some fixed η_0. Nondegeneracy enters through Lemma 2.4 below. To obtain the correct restricted action, we use the condition (3) in Lemma 2.8 below. From then on, we roughly follow the argument used in [3] to prove approximate divisibility. We vary the arrangement slightly to make part of the argument easily available for use elsewhere.

We recall the tracial Rokhlin property for actions of finite cyclic groups on infinite dimensional finite simple unital C*-algebras. The following result is Lemma 1.16 of [28], for the case that the group is cyclic.

Proposition 2.1. Let A be an infinite dimensional finite simple unital C*-algebra, and let $\alpha \in \mathrm{Aut}(A)$ satisfy $\alpha^n = \mathrm{id}_A$. The action of $\mathbb{Z}_n$ generated by α has the tracial Rokhlin property if and only if for every finite set $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_{n-1} \in A$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$, where by convention we take the indices mod n, that is, $e_n = e_0$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and all $a \in F$.
(3) With $e = \sum_{j=0}^{n-1} e_j$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

Definition 2.2. Let θ be a skew symmetric real $d \times d$ matrix. Let

$$n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d \quad \text{and} \quad v = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d} \in A_\theta.$$

We write γ_n for the inner automorphism $\mathrm{Ad}(v)$ of the noncommutative torus A_θ. We further define a homomorphism $\sigma \colon \mathbb{Z}^d \to (S^1)^d$ by the formula $\sigma(n)_j = \exp(2\pi i (\theta n)_j)$ for $1 \leq j \leq d$. (Here the expression θn is the usual action of a $d \times d$ matrix on an element of $\mathbb{R}^d$.)

Lemma 2.3. Let θ be a skew symmetric real $d \times d$ matrix. With γ and σ as in Definition 2.2, we have $\gamma_n(u_j) = \sigma(n)_j u_j$ for $n \in \mathbb{Z}^d$ and $1 \leq j \leq d$. Moreover, if

$m \in \mathbb{Z}^d$, then

$$\gamma_n(u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}) = \exp(2\pi i \langle m, \theta n \rangle) u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$$

for all $n \in \mathbb{Z}^d$.

Proof. The first formula is the special case of the second obtained by setting $m = \delta_j$, the j-th standard basis vector of $\mathbb{Z}^d$. By linearity, both formulas will follow if we check the first when $m = \delta_j$ and $n = \delta_k$. Since $(\theta \delta_k)_j = \theta_{j,k}$, this is just the commutation relation

$$u_k u_j u_k^* = \exp(2\pi i \theta_{j,k}) u_j,$$

which is the same as the one in from Notation 1.1. ∎

Lemma 2.4. Let θ be a skew symmetric real $d \times d$ matrix. The homomorphism $\sigma \colon \mathbb{Z}^d \to (S^1)^d$ of Definition 2.2 has dense range if and only if θ is nondegenerate.

Proof. Assume σ does not have dense range. Let $H = \overline{\sigma(\mathbb{Z}^d)}$, which is a proper closed subgroup of $(S^1)^d$. Choose a nontrivial character $\mu \colon (S^1)^d \to S^1$ whose kernel contains H. By the identification of the dual group of $(S^1)^d$, there is $r \in \mathbb{Z}^d \setminus \{0\}$ such that

$$\mu(\zeta_1, \zeta_2, \ldots, \zeta_d) = \zeta_1^{r_1} \zeta_2^{r_2} \cdots \zeta_d^{r_d}$$

for all $\zeta \in (S^1)^d$. Because $H \subset \operatorname{Ker}(\mu)$, for all $n \in \mathbb{Z}^d$ we have

$$\begin{aligned} 1 = \mu(\sigma(n)) &= \exp(2\pi i (\theta n)_1)^{r_1} \exp(2\pi i (\theta n)_2)^{r_2} \cdots \exp(2\pi i (\theta n)_d)^{r_d} \\ &= \exp(2\pi i \langle r, \theta n \rangle). \end{aligned}$$

Thus θ is degenerate.

Now suppose that θ is degenerate. Then we may choose $r \in \mathbb{Z}^d \setminus \{0\}$ such that $\exp(2\pi i \langle r, \theta n \rangle) = 1$ for all $n \in \mathbb{Z}^d$. Reversing the above calculation, we find that the nontrivial character

$$\mu(\zeta_1, \zeta_2, \ldots, \zeta_d) = \zeta_1^{r_1} \zeta_2^{r_2} \cdots \zeta_d^{r_d}$$

satisfies $\mu(\sigma(n)) = 1$ for all $n \in \mathbb{Z}^d$. Therefore σ does not have dense range. ∎

Corollary 2.5. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $G \subset \mathbb{Z}^d$ be a subgroup with finite index. Let $\sigma \colon \mathbb{Z}^d \to (S^1)^d$ be the homomorphism of Definition 2.2. Then $\sigma(G)$ is dense in $(S^1)^d$.

Proof. Let $H = \overline{\sigma(G)}$. Let S be a set of coset representatives for G in $\mathbb{Z}^d$. Then the sets $\sigma(m)H$, for $m \in S$, are closed and are pairwise equal or disjoint. By Lemma 2.4, their union is $(S^1)^d$. Since there are finitely many of them, and since $(S^1)^d$ is connected, it follows that all are equal to $(S^1)^d$. ∎

Corollary 2.6. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $\zeta_1, \zeta_2, \ldots, \zeta_d \in S^1$. Let $\alpha \in A_\theta$ be the automorphism determined by $\alpha(u_j) = \zeta_j u_j$ for $1 \leq j \leq d$. Then α is approximately inner.

Proof. It suffices to find, for all $\varepsilon > 0$, a unitary $v \in A_\theta$ such that $\|\alpha(u_j) - v u_j v^*\| < \varepsilon$ for $1 \leq j \leq d$. Choose $\delta > 0$ small enough that if $(\omega_1, \omega_2, \ldots, \omega_d) \in (S^1)^d$ satisfies

$$d((\omega_1, \omega_2, \ldots, \omega_d), \, (\zeta_1, \zeta_2, \ldots, \zeta_d)) < \delta,$$

then $|\omega_j - \zeta_j| < \varepsilon$ for $1 \leq j \leq d$. Then use Lemma 2.4 to choose $n \in \mathbb{Z}^d$ such that $d(\sigma(n), \, (\zeta_1, \zeta_2, \ldots, \zeta_d)) < \delta$. Take $v = u_1^{n_1} u_2^{n_2} \cdots u_d^{n_d}$ and use Lemma 2.3. ∎

Lemma 2.7. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n, N \in \mathbb{N}$, and let $1 \leq k \leq d$. Then for every $\varepsilon > 0$ there exists $l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d$ such that:

(1) $v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}$ satisfies $\|v u_j - u_j v\| < \varepsilon$ for $1 \leq j \leq d$.
(2) $l_k = 1 \pmod{n}$.
(3) There is j such that $|l_j| > N$.

Proof. Without loss of generality $k = 1$. Set $\alpha = \mathrm{Ad}(u_1^*)$. There are $\zeta_1, \zeta_2, \ldots, \zeta_d \in S^1$ such that $\alpha(u_j) = \zeta_j u_j$ for $1 \leq j \leq d$. Let $G = n\mathbb{Z} \oplus \mathbb{Z}^{d-1}$, which is a finite index subgroup of $\mathbb{Z}^d$. According to Corollary 2.5, the subgroup $\sigma(G)$ is dense in $(S^1)^d$. Let

$$F = \{l \in \mathbb{Z}^d \colon |l_j| \leq N + 1 \text{ for } 1 \leq j \leq d\}.$$

Since F is finite, $\sigma(G \setminus F)$ is also dense in $(S^1)^d$. Choose $\delta > 0$ small enough that if $(\omega_1, \omega_2, \ldots, \omega_d) \in (S^1)^d$ satisfies

$$d((\omega_1, \omega_2, \ldots, \omega_d),\, (\zeta_1, \zeta_2, \ldots, \zeta_d)) < \delta,$$

then $|\omega_j - \zeta_j| < \varepsilon$ for $1 \leq j \leq d$. Then use density of $\sigma(G \setminus F)$ to choose $r \in G \setminus F$ such that $d(\sigma(r),\, (\zeta_1, \zeta_2, \ldots, \zeta_d)) < \delta$. So with $v_0 = u_1^{r_1} u_2^{r_2} \cdots u_d^{r_d}$, we get $\|v_0 u_j v_0^* - u_1^* u_j u_1\| < \varepsilon$ for $1 \leq j \leq d$. Define

$$l = (r_1 + 1,\, r_2,\, \ldots,\, r_d) \in \mathbb{Z}^d \quad \text{and} \quad v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d} = u_1 v_0 \in A_\theta.$$

Clearly $\|v u_j v^* - u_j\| < \varepsilon$ for $1 \leq j \leq d$. We have $l_1 = 1 \pmod{n}$ because $r_1 \in n\mathbb{Z}$. We have $|l_j| > N$ for some j, because $|r_j| > N + 1$ for some j. ∎

The next lemma is the analog in our context of Lemma 4.6 of [3].

Lemma 2.8. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbb{N}$, let $1 \leq k \leq d$, and let $\eta_0 \in \mathbb{R} \setminus \mathbb{Q}$. Then for every $\varepsilon > 0$ there exist

$$l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d \quad \text{and} \quad m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$$

such that:

(1) $v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}$ and $w = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ satisfy $\|v u_j - u_j v\| < \varepsilon$ and $\|w u_j - u_j w\| < \varepsilon$ for $1 \leq j \leq d$.
(2) There is $\eta \in \mathbb{R} \setminus \mathbb{Q}$ such that $|\exp(2\pi i \eta) - \exp(2\pi i \eta_0)| < \varepsilon$ and the unitaries v and w of Part (1) satisfy $wv = \exp(2\pi i \eta) vw$.
(3) $l_k = 1 \pmod{n}$ and $m_k = 0 \pmod{n}$.

Proof. Without loss of generality $k = 1$ and $\eta_0 \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. Choose $N \in \mathbb{N}$ so large that $2\pi/N < \varepsilon$. Use Lemma 2.7 with θ, n, and ε as given, with $k = 1$, and with this value of N, obtaining

$$l \in \mathbb{Z}^d \quad \text{and} \quad v = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}.$$

Note in particular that $\|v u_j v^* - u_j\| < \varepsilon$ for $1 \leq j \leq d$ and $l_1 = 1 \pmod{n}$. Let s be an index such that $|l_s| > N$.

Let

$$T = \left\{\eta \in \mathbb{R} \colon (u_1^{r_1} u_2^{r_2} \cdots u_d^{r_d})\, v\, (u_1^{r_1} u_2^{r_2} \cdots u_d^{r_d})^* = \exp(2\pi i \eta) v \text{ for some } r \in \mathbb{Z}^d\right\}.$$

Then T is a subgroup of $\mathbb{R}$ which is generated by $d + 1$ elements, namely 1 and elements corresponding to letting r run through the standard basis vectors of $\mathbb{Z}^d$. So $T \cap \mathbb{Q}$ is also finitely generated, and is therefore discrete. Since $\eta_0 \notin \mathbb{Q}$, we have $\mathrm{dist}(\eta_0,\, T \cap \mathbb{Q}) > 0$. Set $\varepsilon_0 = \min(\varepsilon,\, \mathrm{dist}(\eta_0,\, T \cap \mathbb{Q}))$.

Set

$$M = \sum_{j=1}^{d} |l_j| \quad \text{and} \quad \delta = \min\left(\tfrac{1}{2}\varepsilon_0,\, M^{-1}\varepsilon_0\right).$$

Let G be the finite index subgroup $G = n\mathbb{Z} \oplus \mathbb{Z}^{d-1} \subset \mathbb{Z}^d$. Let

$$\lambda = (1,\, \ldots,\, 1,\, \exp(2\pi i \eta_0/l_s),\, 1,\, \ldots,\, 1) \in (S^1)^d,$$

where $\exp(2\pi i \eta_0/l_s)$ is in position s. Use Corollary 2.5 and Lemma 2.3 to choose $m \in G$ such that $\sigma(m)$, as in Definition 2.2, is so close to λ that $w = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ satisfies $\|w u_j w^* - u_j\| < \delta$ for $j \neq s$, and $\|w u_s w^* - \exp(2\pi i \eta_0/l_s) u_s\| < \delta$.

Since $\delta \leq \varepsilon$, it is clear that $\|w u_j w^* - u_j\| < \varepsilon$ for $j \neq s$. Also

$$\|w u_s w^* - u_s\| \leq \|w u_s w^* - \exp(2\pi i \eta_0/l_s) u_s\| + |\exp(2\pi i \eta_0/l_s) - 1|.$$

Using $\delta \leq \frac{1}{2}\varepsilon$, the first term is less than $\frac{1}{2}\varepsilon$. The second term satisfies

$$|\exp(2\pi i \eta_0/l_s) - 1| < 2\pi \left|\frac{\eta_0}{l_s}\right| < 2\pi \left(\frac{1}{2N}\right) \leq \tfrac{1}{2}\varepsilon.$$

Therefore $\|w u_j w^* - u_j\| < \varepsilon$ for $j = s$ as well. This completes the verification of Part (1) of the conclusion. Part (3) holds because $m_1 \in n\mathbb{Z}$ by construction.

It remains to prove Part (2). For each j with $1 \leq j \leq d$, there is $\zeta_j \in S^1$ such that $w u_j w^* = \zeta_j u_j$. Then

$$w v w^* = \zeta_1^{l_1} \zeta_2^{l_2} \cdots \zeta_d^{l_d} v.$$

Thus $wv = \exp(2\pi i \eta) v w$ for some $\eta \in \mathbb{R}$. By construction we have $|\zeta_j - 1| < M^{-1}\varepsilon_0$ for $j \neq s$, and $|\zeta_s - \exp(2\pi i \eta_0/l_s)| < M^{-1}\varepsilon_0$. It follows that

$$\begin{aligned}\left|\zeta_1^{l_1} \zeta_2^{l_2} \cdots \zeta_d^{l_d} - \exp(2\pi i \eta_0/l_s)^{l_s}\right| &\leq |l_s| \cdot |\zeta_s - \exp(2\pi i \eta_0/l_s)| + \sum_{j \neq s} |l_j| \cdot |\zeta_j - 1| \\ &< \sum_{j=1}^{d} |l_j| M^{-1} \varepsilon_0 \leq \varepsilon_0.\end{aligned}$$

Therefore

$$\|wv - \exp(2\pi i \eta_0) v w\| = \left|\zeta_1^{l_1} \zeta_2^{l_2} \cdots \zeta_d^{l_d} - \exp(2\pi i \eta_0)\right| < \varepsilon_0,$$

which is the same as $|\exp(2\pi i \eta) - \exp(2\pi i \eta_0)| < \varepsilon_0$. In particular, $|\exp(2\pi i \eta) - \exp(2\pi i \eta_0)| < \varepsilon$, as desired. Moreover, $\eta \in T$ and there is no $\rho \in T \cap \mathbb{Q}$ such that $|\exp(2\pi i \rho) - \exp(2\pi i \eta_0)| < \varepsilon_0$, whence $\eta \notin \mathbb{Q}$. ∎

The proofs of the next two results together parallel the proof of Theorem 1.5 of [3]. The first of them says, roughly, that higher dimensional noncommutative tori contain approximately central copies of irrational rotation algebras, constructed in a special way. Unfortunately, the rotation parameter varies with the degree of approximation.

Lemma 2.9. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, let $n \in \mathbb{N}$, and let $1 \leq k \leq d$. Then for every $\eta_0 \in \mathbb{R}$, every open set $U \subset S^1$ containing $\exp(2\pi i \eta_0)$, every finite subset $F \subset A_\theta$, every finite subset $S \subset \Gamma(U)$ (following Notation 1.11), and every $\varepsilon > 0$, there exist $\eta \in \mathbb{R} \setminus \mathbb{Q}$ and

$$l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d \quad \text{and} \quad m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$$

such that:

(1) $|\eta - \eta_0| < \varepsilon$ and $\exp(2\pi i \eta) \in U$.
(2) $x = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d}$ and $y = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}$ satisfy $yx = \exp(2\pi i \eta) xy$.
(3) Following the notation of Theorem 1.10, and with x and y as in Part (2), let $\varphi \colon A_\eta \to A_\theta$ be the homomorphism such that $\varphi(v_\eta) = x$ and $\varphi(w_\eta) = y$. Then $\| [a, \, \varphi(b(\eta))] \| < \varepsilon$ for all $a \in F$ and all $b \in S$.
(4) $l_k = 1 \pmod{n}$ and $m_k = 0 \pmod{n}$.

Proof. Let the notation be as in Theorem 1.10 and Notation 1.11.

Without loss of generality $\varepsilon < 1$. Then there is $\varepsilon_0 > 0$ such that whenever $\zeta \in S^1$ satisfies $|\zeta - \exp(2\pi i \eta_0)| < \varepsilon_0$, there is a unique $\eta \in \mathbb{R}$ such that $\exp(2\pi i \eta) = \zeta$ and $|\eta - \eta_0| < \varepsilon$.

Without loss of generality $\|a\| \leq 1$ for all $a \in F$. Replacing U by an open set V with $\exp(2\pi i \eta_0) \in V \subset \overline{V} \subset U$, we may assume every $b \in S$ is bounded. Then without loss of generality $\|b(\eta)\| \leq 1$ for all $b \in S$ and $\eta \in U$. Write $F = \{a_1, a_2, \ldots, a_s\}$ and $S = \{b_1, b_2, \ldots, b_t\}$. Choose polynomials $g_1, g_2, \ldots, g_t$ in four noncommuting variables such that

$$\| g_r(v_{\eta_0}, \, v_{\eta_0}^*, \, w_{\eta_0}, \, w_{\eta_0}^*) - b_r(\eta_0) \| < \tfrac{1}{7} \varepsilon$$

for $1 \leq r \leq t$. Because the rotation algebras form a continuous field over S^1 (Theorem 1.10), there is $\delta > 0$ such that whenever $|\eta - \eta_0| < \delta$ we have $\exp(2\pi i \eta_0) \in U$, and

$$\| g_r(v_\eta, \, v_\eta^*, \, w_\eta, \, w_\eta^*) - b_r(\eta) \| < \tfrac{2}{7} \varepsilon$$

for $1 \leq r \leq t$.

Choose polynomials $f_1, f_2, \ldots, f_t$ in $2d$ noncommuting variables such that

$$\| f_r(u_1, \, u_1^*, \, \ldots, \, u_d, \, u_d^*) - a_r \| < \frac{\varepsilon}{7(1+\varepsilon)}$$

for $1 \leq r \leq s$. Choose (see Proposition 4.3 of [3]) $\delta_0 > 0$ such that whenever D is a C*-algebra and

$$c_1, c_2, \ldots, c_{2d}, d_1, d_2, d_3, d_4 \in D$$

are elements of norm 1 which satisfy $\| [c_r, d_j] \| < \delta_0$ for all j and r, then

$$\| [f_r(c_1, c_2, \ldots, c_{2d}), \, g_j(d_1, d_2, d_3, d_4)] \| < \tfrac{1}{7} \varepsilon$$

for $1 \leq r \leq s$ and $1 \leq j \leq t$.

Apply Lemma 2.8 with θ, n, η_0, and k as given, and with $\min(\varepsilon_0, \delta, \delta_0)$ in place of ε. We obtain $\eta \in \mathbb{R} \setminus \mathbb{Q}$ and

$$l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d \quad \text{and} \quad m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d.$$

Set

$$x = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d} \quad \text{and} \quad y = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d}.$$

By the choice of ε_0, we may assume that $|\eta - \eta_0| < \varepsilon$, and by the choice of δ we have $\exp(2\pi i \eta_0) \in U$. This is Part (1) of the conclusion. Parts (2) and (4) are immediate.

It remains to prove Part (3). Part (1) of the conclusion of Lemma 2.8 and the choice of δ_0 ensure that

$$\| [f_r(u_1, \, u_1^*, \, \ldots, \, u_d, \, u_d^*), \, g_j(x, x^*, y, y^*)] \| < \tfrac{1}{7} \varepsilon$$

for $1 \leq r \leq s$ and $1 \leq j \leq t$. From the choice of δ, we get

$$\| g_j(x, x^*, y, y^*) \| < \| \varphi(b_j(\eta)) \| + \tfrac{2}{7} \varepsilon < 1 + \varepsilon$$

for $1 \leq j \leq t$. Using the choice of the polynomials f_r, we therefore get

$$\begin{aligned} \|[a_r,\, \varphi(b_j(\eta))]\| &\leq 2\|a_r\| \cdot \|\varphi(b_j(\eta)) - g_j(x, x^*, y, y^*)\| \\ &\qquad + 2\|a_r - f_r(u_1,\, u_1^*,\, \ldots,\, u_d,\, u_d^*)\| \cdot \|g_j(x, x^*, y, y^*)\| \\ &\qquad + \|[f_r(u_1,\, u_1^*,\, \ldots,\, u_d,\, u_d^*),\, g_j(x, x^*, y, y^*)]\| \\ &< 2\left(\frac{2\varepsilon}{7}\right) + 2(1+\varepsilon)\left(\frac{\varepsilon}{7(1+\varepsilon)}\right) + \frac{\varepsilon}{7} = \varepsilon \end{aligned}$$

for $1 \leq r \leq s$ and $1 \leq j \leq t$, as desired. ∎

Proposition 2.10. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbb{N}$, let $\omega = \exp(2\pi i/n)$, let $1 \leq k \leq d$, and, following Notation 1.1, let $\alpha\colon A_\theta \to A_\theta$ the unique automorphism satisfying $\alpha(u_k) = \omega u_k$ and $\alpha(u_r) = u_r$ for $r \neq k$. Then the action of $\mathbb{Z}_n$ generated by α has the tracial Rokhlin property.

Proof. Let τ be the unique tracial state on A_θ (Theorem 1.9). We will show that for every $\varepsilon > 0$ and every finite subset $F \subset A_\theta$, there are mutually orthogonal projections $e_0, e_1, \ldots, e_{n-1} \in A_\theta$ such that:

(1) $\|\alpha(e_j) - e_{j+1}\| < \varepsilon$ for $0 \leq j \leq n-1$.
(2) $\|e_j a - a e_j\| < \varepsilon$ for $0 \leq j \leq n-1$ and $a \in F$.
(3) $1 - n\tau(e_0) < \varepsilon$.

We first argue that this is enough to deduce the tracial Rokhlin property. We must prove Condition (3) in Proposition 2.1. We first recall (see Theorems 6.1 and 7.1 of [32], or Theorems 1.4(d) and 1.5 of [3]) that if $p, q \in A_\theta$ are projections with $\tau(p) < \tau(q)$, then $p \precsim q$. Also, A_θ has Property (SP) by Theorem 1.4(b) of [3]. If now a nonzero positive element $x \in A_\theta$ is given, then we may use Property (SP) to find a nonzero projection $p \in \overline{xAx}$. Require $\varepsilon \leq \min(\tau(p), 1)$. Then $\alpha(e_j) \sim e_{j+1}$. Let $e = \sum_{j=0}^{n-1} e_j$. This gives $\tau(1-e) = 1 - n\tau(e_0) < \varepsilon$, whence $\tau(e_0) > \frac{1}{n}(1-\varepsilon)$. Now $\tau(1-e) < \varepsilon$, which implies $\tau(1-e) < \tau(p)$, so that Condition (3) of Proposition 2.1 follows from the comparison result above.

Now we prove Conditions (1), (2), and (3) at the beginning of the proof. Let the notation be as in Theorem 1.10 and Notation 1.11. Let $\varepsilon > 0$. Choose and fix $\eta_0 \in \mathbb{R} \setminus \mathbb{Q}$. Choose $\varepsilon_1 > 0$ such that whenever $a_0, a_1, \ldots, a_{n-1}$ are elements of a unital C*-algebra D with

$$\|a_j a_r - \delta_{j,r} a_j\| < \varepsilon_1 \quad \text{and} \quad \|a_j^* - a_j\| < \varepsilon_1$$

for $0 \leq j,\, r \leq n-1$, then there are mutually orthogonal projections

$$q_0, q_1, \ldots, q_{n-1} \in D$$

such that $\|q_j - a_j\| < \frac{1}{3} n^{-1} \varepsilon$ for $0 \leq j \leq n-1$. (For example, apply Definition 2.2 and Lemma 2.3 of [3] with the finite dimensional C*-algebra B taken to be $\mathbb{C}^{n+1}$, using in addition the element $a_n = 1 - \sum_{j=0}^{n-1} a_j$.) Let $p_0, p_1, \ldots, p_{n-1} \in A_{\eta_0}$ be the projections $e_0, e_1, \ldots, e_{n-1}$ of Lemma 1.13 for η_0 in place of η and $\frac{1}{3}\varepsilon$ in place of ε. Because the rotation algebras form a continuous field over S^1 with section algebra A (Theorem 1.10), we may choose $c_0, c_1, \ldots, c_{n-1} \in A$ such that $\mathrm{ev}_{\eta_0}(c_j) = p_j$ for $0 \leq j \leq n-1$, and we can furthermore find $\delta_0 > 0$ such that $|\exp(2\pi i\eta) - \exp(2\pi i\eta_0)| < \delta_0$ implies

$$\|\mathrm{ev}_\eta(c_j)\mathrm{ev}_\eta(c_r) - \delta_{j,r}\mathrm{ev}_\eta(c_j)\| < \varepsilon_1 \quad \text{and} \quad \|\mathrm{ev}_\eta(c_j)^* - \mathrm{ev}_\eta(c_j)\| < \varepsilon_1$$

for $0 \leq j,\, r \leq n-1$. Let $V \subset S^1$ be an open set such that $\exp(2\pi i \eta_0) \in V$ and such that $\zeta \in \overline{V}$ implies $|\zeta - \exp(2\pi i \eta_0)| < \delta_0$. Letting $c_j|_{\overline{V}}$ denote the restriction of c_j, regarded as a section, to $\overline{V}$, we get

$$\|(c_j|_{\overline{V}})(c_r|_{\overline{V}}) - \delta_{j,r} c_j|_{\overline{V}}\| < \varepsilon_1 \quad \text{and} \quad \left\|(c_j|_{\overline{V}})^* - c_j|_{\overline{V}}\right\| < \varepsilon_1$$

for $0 \leq j,\, r \leq n-1$, so that there are mutually orthogonal projections

$$q_0, q_1, \ldots, q_{n-1} \in \Gamma(\overline{V})$$

such that $\|q_j - c_j|_{\overline{V}}\| < \frac{1}{3} n^{-1} \varepsilon$ for $0 \leq j \leq n-1$. Since the restriction map $A = \Gamma(S^1) \to \Gamma(\overline{V})$ is surjective, there exist $b_0, b_1, \ldots, b_{n-1} \in A$ such that $b_j|_{\overline{V}} = q_j$ for $0 \leq j \leq n-1$.

Let the generators of A be as in Theorem 1.10, and let $\beta \in \mathrm{Aut}(A)$ be the unique automorphism such that

$$\beta(v) = \omega v, \quad \beta(w) = w, \quad \text{and} \quad \beta(z) = z.$$

Let $\beta_\eta \in \mathrm{Aut}(A_\eta)$ be defined by $\beta_\eta(v_\eta) = \omega v_\eta$ and $\beta_\eta(w_\eta) = w_\eta$. Then $\mathrm{ev}_\eta \circ \beta = \beta_\eta \circ \mathrm{ev}_\eta$. Since β sends continuous sections to continuous sections, there is an open set $U_0 \subset V$ such that $\eta_0 \in U_0$ and if $\eta \in U_0$ then for $0 \leq j \leq n-1$ and with $b_n = b_0$,

$$\|\beta_\eta(b_j(\eta)) - b_{j+1}(\eta)\| \quad \text{and} \quad \|\beta_{\eta_0}(b_j(\eta_0)) - b_{j+1}(\eta_0)\|$$

differ by less than $\frac{1}{3}\varepsilon$. For such η we have $b_j(\eta) = q_j(\eta)$, so, using $c_j(\eta_0) = p_j$ and $\beta_{\eta_0}(p_j) = p_{j+1}$ at the second last step,

$$\begin{aligned}
\|\beta_\eta(q_j(\eta)) - q_{j+1}(\eta)\| &< \|\beta_{\eta_0}(q_j(\eta_0)) - q_{j+1}(\eta_0)\| + \tfrac{1}{3}\varepsilon \\
&< \|q_j - c_j|_{\overline{V}}\| + \|q_{j+1} - c_{j+1}|_{\overline{V}}\| + \|\beta_{\eta_0}(c_j(\eta_0)) - c_{j+1}(\eta_0)\| + \tfrac{1}{3}\varepsilon \\
&< \tfrac{1}{3} n^{-1}\varepsilon + \tfrac{1}{3} n^{-1}\varepsilon + \tfrac{1}{3}\varepsilon \leq \varepsilon
\end{aligned}$$

for $0 \leq j \leq n-1$.

Using Lemma 1.12, choose an open set $U \subset U_0$ such that $\eta_0 \in U$ and if $\eta \in U$ then for $0 \leq j \leq n-1$ we have $|\tau_\eta(q_j(\eta)) - \tau_{\eta_0}(q_j(\eta_0))| < \frac{1}{3} n^{-1}\varepsilon$.

Apply Lemma 2.9 with θ, n, k, η_0, U, and F as given, with $\min(\varepsilon, \delta)$ in place of ε, and with $S = \{q_0, q_1, \ldots, q_{n-1}\}$. We obtain $\eta \in (\mathbb{R} \setminus \mathbb{Q}) \cap U$ and

$$l = (l_1, l_2, \ldots, l_d) \in \mathbb{Z}^d \quad \text{and} \quad m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d.$$

Set

$$x = u_1^{l_1} u_2^{l_2} \cdots u_d^{l_d} \quad \text{and} \quad y = u_1^{m_1} u_2^{m_2} \cdots u_d^{m_d},$$

so that $yx = \exp(2\pi i \eta) xy$. Let $\varphi\colon A_\eta \to A_\theta$ be the homomorphism such that $\varphi(v_\eta) = x$ and $\varphi(w_\eta) = y$, and set $e_j = \varphi(q_j(\eta))$ for $0 \leq j \leq n-1$. We verify Conditions (1), (2), and (3) at the beginning of the proof for this choice of $e_0, e_1, \ldots, e_{n-1}$.

We do Condition (1). Because $l_k = 1 \pmod{n}$ and $m_k = 0 \pmod{n}$, we have $\alpha(x) = \omega x$ and $\alpha(y) = y$. It follows that $\alpha \circ \varphi = \varphi \circ \beta_\eta$. Therefore

$$\|\alpha(e_j) - e_{j+1}\| \leq \|\beta_\eta(q_j(\eta)) - q_{j+1}(\eta)\| < \varepsilon$$

for $0 \leq j \leq n-1$, as desired.

Condition (2) is immediate from Part (3) of Lemma 2.9.

Finally, we check Condition (3). By uniqueness of the tracial states, we have $\tau \circ \varphi = \tau_\eta$. Therefore, using the choice of U at the second step and $\|q_j(\eta_0) - p_j\| < \frac{1}{3}n^{-1}\varepsilon$ at the third step, we get

$$\tau(e_j) = \tau_\eta(q_j(\eta)) > \tau_{\eta_0}(q_j(\eta_0)) - \tfrac{1}{3}n^{-1}\varepsilon > \tau_{\eta_0}(p_j) - \tfrac{2}{3}n^{-1}\varepsilon.$$

Therefore

$$1 - n\tau(e_0) < 1 - n\tau(p_0) + \tfrac{2}{3}\varepsilon < \tfrac{1}{3}\varepsilon + \tfrac{2}{3}\varepsilon = \varepsilon.$$

This completes the proof of (3). ∎

We next need to identify the fixed point algebra of the action in Proposition 2.10 with a suitable higher dimensional noncommutative torus. The following lemma, suggested by Hanfeng Li, is a substantial generalization of our original statement.

Lemma 2.11. Let θ be a skew symmetric real $d \times d$ matrix. Let

$$M = (m_{j,k})_{1 \leq j,k \leq d} \in \mathrm{GL}_d(\mathbb{R}) \cap M_n(\mathbb{Z}),$$

and set $\widetilde{\theta} = M^{\mathrm{t}} \theta M$. Let $u_1, u_2, \ldots, u_d$ be the standard generators of A_θ (as in Notation 1.1), and let $\widetilde{u}_1, \widetilde{u}_2, \ldots, \widetilde{u}_d$ be the standard generators of $A_{\widetilde{\theta}}$. For $1 \leq k \leq d$, define

$$v_k = u_1^{m_{1,k}} \cdot u_2^{m_{2,k}} \cdots u_d^{m_{d,k}} \in A_\theta.$$

Then $\widetilde{u}_k \mapsto v_k$ extends to an isomorphism $A_{\widetilde{\theta}} \to C^*(v_1, v_2, \ldots, v_d)$.

Proof. It is easy to check that there exists a homomorphism $\varphi \colon A_{\widetilde{\theta}} \to A_\theta$ such that $\varphi(\widetilde{u}_k) = v_k$ for $1 \leq k \leq d$, and clearly $\varphi(A_{\widetilde{\theta}}) = C^*(v_1, v_2, \ldots, v_d)$. We need only check that φ is injective.

Define a group action $\gamma \colon (S^1)^d \to \mathrm{Aut}(A_\theta)$ by $\gamma_{\zeta_1, \zeta_2, \ldots, \zeta_d}(u_k) = \zeta_k u_k$ for $1 \leq k \leq d$ and $\zeta_1, \zeta_2, \ldots, \zeta_d \in S^1$. Similarly define $\widetilde{\gamma} \colon (S^1)^d \to \mathrm{Aut}(A_{\widetilde{\theta}})$. It is well known (and is easily checked by averaging monomials in the u_k and $\widetilde{u}_k$ over the group) that the fixed point algebras A_θ^γ and $A_{\widetilde{\theta}}^{\widetilde{\gamma}}$ are both $\mathbb{C} \cdot 1$. In particular, the restriction of φ to $A_{\widetilde{\theta}}^{\widetilde{\gamma}}$ is injective.

Since $M \in M_d(\mathbb{Z})$, we have $M^{\mathrm{t}} \mathbb{Z}^d \subset \mathbb{Z}^d$, so that M^{t} descends to a homomorphism $f \colon (S^1)^d \to (S^1)^d$. This homomorphism is surjective because M is invertible over $\mathbb{R}$. Define $\beta \colon (S^1)^d \to \mathrm{Aut}(A_{\widetilde{\theta}})$ by $\beta_{\zeta_1, \zeta_2, \ldots, \zeta_d} = \widetilde{\gamma}_{f(\zeta_1, \zeta_2, \ldots, \zeta_d)}$. Then one checks, by examining the generators $\widetilde{u}_k$, that $\varphi \circ \beta_{\zeta_1, \zeta_2, \ldots, \zeta_d} = \gamma_{\zeta_1, \zeta_2, \ldots, \zeta_d} \circ \varphi$ for $\zeta_1, \zeta_2, \ldots, \zeta_d \in S^1$. Injectivity of $\varphi|_{A_{\widetilde{\theta}}^{\widetilde{\gamma}}}$ now implies injectivity of φ, since if a is a nonzero positive element of $\mathrm{Ker}(\varphi)$, then averaging over $(S^1)^d$ gives a nonzero positive element of $\mathrm{Ker}(\varphi) \cap A_{\widetilde{\theta}}^{\widetilde{\gamma}}$. ∎

Corollary 2.12. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $n \in \mathbb{N}$, let $1 \leq l \leq d$, and let

$$B = \mathrm{diag}(1, \ldots, 1, n, 1, \ldots, 1) \in \mathrm{GL}_d(\mathbb{Q}),$$

where n is in the l-th position. Then $A_{B^{\mathrm{t}} \theta B}$ has tracial rank zero if and only if A_θ has tracial rank zero.

Proof. With u_k as in Notation 1.1, set

$$D = C^*(u_1, \, \ldots, \, u_{l-1}, \, u_l^n, \, u_{l+1}, \, \ldots, \, u_d) \subset A_\theta.$$

Calculating $B^{\mathrm{t}} \theta B$, we find that $D \cong A_{B^{\mathrm{t}} \theta B}$ by Lemma 2.11.

Let $\alpha\colon A_\theta \to A_\theta$ the unique automorphism satisfying $\alpha(u_l) = \exp(2\pi i/n)u_l$ and $\alpha(u_k) = u_k$ for $k \neq l$. We claim that $A_\theta^\alpha = D$. That $D \subset A_\theta^\alpha$ is clear. For the reverse inclusion, define $E\colon A_\theta \to A_\theta^\alpha$ by $E(a) = \frac{1}{n}\sum_{j=0}^{n-1}\alpha^j(a)$. Then E is a surjective continuous linear map, so it suffices to show that $E\big(u_1^{m_1}u_2^{m_2}\cdots u_d^{m_d}\big) \in D$ for all $m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$. If m_l is divisible by n then $u_1^{m_1}u_2^{m_2}\cdots u_d^{m_d}$ is a fixed point of E and is in D, and otherwise $E\big(u_1^{m_1}u_2^{m_2}\cdots u_d^{m_d}\big) = 0 \in D$. This proves the claim.

Using Proposition 2.10 and Corollary 2.6, the result now follows from Theorem 4.8 of [28]. ∎

3. Direct limit decomposition for simple noncommutative tori

In this section, we use the results of the previous two sections to prove that every simple higher dimensional noncommutative torus is an AT algebra.

The following result is essentially Corollary 6.6 of [18].

Proposition 3.1. Let α be a nondegenerate skew symmetric real bicharacter on $\mathbb{Z}^n$. Suppose that $A_{\alpha|_{\mathbb{Z}^{n-1}\times\{0\}}}$ is a simple AT algebra with real rank zero. Then A_α is a simple AT algebra with real rank zero.

Proof. Let $\beta = \alpha|_{\mathbb{Z}^{n-1}\times\{0\}}$. We note that $K_0(A_\beta) \cong K_1(A_\beta) \cong \mathbb{Z}^{2^{n-1}}$ by Lemma 1.5 and by repeated application of the Pimsner-Voiculescu exact sequence [31]. In particular, both groups are finitely generated. Further write $A_\alpha = C^*(\mathbb{Z}, A_\beta, \varphi)$ as in Lemma 1.5, with φ homotopic to the identity. Thus, in the notation of [18] (see the introduction to [18]), $\varphi \in \mathrm{HInn}(A_\beta)$. So the proof of Corollary 6.5 of [18] shows that the hypotheses of Theorem 6.4 of [18] hold. We know from Lemma 1.9 that $A_\alpha = C^*(\mathbb{Z}, A_\beta, \varphi)$ has a unique tracial state. Therefore Theorem 6.4 of [18] implies that $A_\alpha = C^*(\mathbb{Z}, A_\beta, \varphi)$ is a simple AT algebra with real rank zero. ∎

It is worth pointing out another version of the proof. The automorphism φ has the Rokhlin property by Theorem 6.1 of [17] and the preceding remark in [17]. The algebra A_β has tracial rank zero, since it is a simple unital AT algebra with real rank zero. Therefore φ has the tracial Rokhlin property for actions of $\mathbb{Z}$, by Theorem 1.12 of [26]. It is approximately inner by Corollary 2.6. So Theorem 3.9 of [24] applies, showing that A_α has tracial rank zero. Now one can conclude that A_α is an AT algebra by the same proof as for Theorem 3.8. However, tracial rank zero is all that is really needed here.

Lemma 3.2. The group $\mathrm{GL}_d(\mathbb{Q})$ is generated as a group by $\mathrm{GL}_d(\mathbb{Z})$ and all matrices of the form $\mathrm{diag}(1, \ldots, 1, n, 1, \ldots, 1)$, where $n \in \mathbb{N}$ is nonzero and is in an arbitrary position.

Proof. Let G be the subgroup of $\mathrm{GL}_d(\mathbb{Q})$ generated by $\mathrm{GL}_d(\mathbb{Z})$ and the matrices $\mathrm{diag}(1, \ldots, 1, n, 1, \ldots, 1)$. It suffices to show that G contains all of the following three kinds of elementary matrices:

$$E_j^{(1)}(r) = \mathrm{diag}(1, \ldots, 1, r, 1, \ldots, 1),$$

where $r \in \mathbb{Q} \setminus \{0\}$ and is the j-th diagonal entry in the matrix; the transposition matrix $E_{j,k}^{(2)}$, for $1 \leq j < k \leq d$, which acts on the standard basis vectors by

$$E_{j,k}^{(2)}(\delta_l) = \begin{cases} \delta_l & l \neq j,\, k \\ \delta_k & l = j \\ \delta_j & l = k; \end{cases}$$

and the matrix $E_{j,k}^{(3)}(r)$ for $1 \leq j,\, k \leq n$ with $j \neq k$ and $r \in \mathbb{Q}$, given by

$$E_{j,k}^{(3)}(r)(\delta_l) = \begin{cases} \delta_l & l \neq k \\ \delta_k + r\delta_j & l = k. \end{cases}$$

If $r = (-1)^m p/q$ with $m = 0$ or $m = 1$ and with p and q positive integers, then

$$E_j^{(1)}(r) = E_j^{(1)}((-1)^m) E_j^{(1)}(p) \big[E_j^{(1)}(q) \big]^{-1},$$

where the first factor is in $\mathrm{GL}_d(\mathbb{Z})$ and $E_j^{(1)}(p)$ and $E_j^{(1)}(q)$ are among the other generators of G. The matrix $E_{j,k}^{(2)}$ is already in $\mathrm{GL}_d(\mathbb{Z})$. For $E_{j,k}^{(3)}(r)$, we may conjugate by a permutation matrix, which is in $\mathrm{GL}_d(\mathbb{Z})$, and split off as a direct summand a $(d-2) \times (d-2)$ identity matrix, and thus reduce to the case $d = 2$, $j = 1$, and $k = 2$. Write $r = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then the factorization

$$E_{1,2}^{(3)}(r) = \begin{pmatrix} 1 & p/q \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$$

shows that $E_{1,2}^{(3)}(r) \in G$. ∎

Corollary 3.3. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Let $B \in \mathrm{GL}_d(\mathbb{Q})$. Then $A_{B^{\mathrm{t}}\theta B}$ has tracial rank zero if and only if A_θ has tracial rank zero.

Proof. Combine Lemma 3.2, Remark 1.2, and Corollary 2.12. ∎

Lemma 3.4. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d > 2$. Suppose that there is no subgroup H of $\mathbb{Z}^d$ of rank $d-1$ such that $\theta|_H$ (in the sense of Remark 1.4) is nondegenerate. Let $r < d-1$ be the maximal rank of a proper subgroup H of $\mathbb{Z}^d$ such that $\theta|_H$ is nondegenerate. Then there exists $B \in \mathrm{GL}_d(\mathbb{Q})$ such that $B^{\mathrm{t}}\theta B$ has the block form

$$B^{\mathrm{t}}\theta B = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ (\rho_{1,2})^{\mathrm{t}} & \rho_{2,2} \end{pmatrix},$$

with $\rho_{1,1}$ and $\rho_{2,2}$ nondegenerate skew symmetric real $r \times r$ and $(d-r) \times (d-r)$ matrices, and where all the entries of $\rho_{1,2}$ are in $\mathbb{Z}$.

Proof. Let $H \subset \mathbb{Z}^d$ be a subgroup of rank r such that $\theta|_H$ is nondegenerate. Since θ is not rational, clearly $r \geq 2$. Let $(v_1, v_2, \ldots, v_r)$ be a basis for H over $\mathbb{Z}$. Choose $v_{r+1}, \ldots, v_d \in \mathbb{Z}^d$ such that $(v_1, v_2, \ldots, v_d)$ is a basis for $\mathbb{Q}^d$ over $\mathbb{Q}$. For $r+1 \leq k \leq d$, by hypothesis $\theta|_{H + \mathbb{Z}v_k}$ is degenerate. By Lemma 1.7, there exists $x_k \in \mathrm{span}_{\mathbb{Q}}(H \cup \{v_k\}) \setminus \{0\}$ such that $\langle x_k, \theta y \rangle \in \mathbb{Q}$ for all $y \in \mathrm{span}_{\mathbb{Q}}(H \cup \{v_k\})$. Since $\theta|_H$ is nondegenerate, we have $x_k \notin \mathrm{span}_{\mathbb{Q}}(H)$. Therefore $v_k \in \mathrm{span}_{\mathbb{Q}}(H \cup \{x_k\})$. It follows that

$$v_{r+1}, \ldots, v_d \in \mathrm{span}_{\mathbb{Q}}(v_1, v_2, \ldots, v_r, x_{r+1}, \ldots, x_d),$$

so that $(v_1, v_2, \ldots, v_r, x_{r+1}, \ldots, x_d)$ is a basis for $\mathbb{Q}^d$. By construction, we have $\langle x_k, \theta v_l \rangle \in \mathbb{Q}$ for $1 \leq l \leq r$ and $r+1 \leq k \leq d$. Choose $N \in \mathbb{Z} \setminus \{0\}$ such that $N\langle x_k, \theta v_l \rangle \in \mathbb{Z}$ for $1 \leq l \leq r$ and $r+1 \leq k \leq d$.

Let $B \in \mathrm{GL}_d(\mathbb{Q})$ be the matrix whose action on the standard basis vectors is

$$B\delta_k = \begin{cases} v_k & 1 \leq k \leq r \\ Nx_k & r+1 \leq k \leq d. \end{cases}$$

Then for $1 \leq l \leq r$ and $r+1 \leq k \leq d$, we have

$$\langle \delta_k, B^{\mathrm{t}} \theta B \delta_l \rangle = N\langle x_k, \theta v_l \rangle \in \mathbb{Z}.$$

Since $B^{\mathrm{t}}\theta B$ is skew symmetric, this shows that it has a block decomposition of the required form. It is immediate to check that the two diagonal blocks must be nondegenerate, since otherwise $B^{\mathrm{t}}\theta B$ would be degenerate, contradicting Lemma 1.8. ∎

Theorem 3.5. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 2$. Then A_θ has tracial rank zero.

Proof. We prove this by induction on d. For $d = 2$, the Elliott-Evans Theorem [10] shows that A_θ is a simple AT algebra with real rank zero, and tracial rank zero then follows from Proposition 2.6 of [19] (with $\mathcal{C}$ as defined in 2.5 of [19]). Suppose d is given, and the theorem is known for all skew symmetric real $k \times k$ matrices with $k < d$. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. There are two cases.

First, suppose that there is a subgroup H_0 of $\mathbb{Z}^d$ of rank $d-1$ such that $\theta|_{H_0}$ (in the sense of Remark 1.4) is nondegenerate. Set

$$H = \{x \in \mathbb{Z}^d \colon \text{There is } n \in \mathbb{Z} \text{ such that } nx \in H_0\}.$$

Then H is also a subgroup of $\mathbb{Z}^d$ of rank $d-1$, and $\theta|_{H_0}$ is also nondegenerate. Moreover, $\mathbb{Z}^d/H$ is torsion free and therefore isomorphic to $\mathbb{Z}$, from which it follows that the quotient map splits. Thus there is an isomorphism $\mathbb{Z}^d \to \mathbb{Z}^d$ which sends H isomorphically onto $\mathbb{Z}^{d-1} \oplus \{0\} \subset \mathbb{Z}^d$. Accordingly, we may assume that $H = \mathbb{Z}^{d-1} \oplus \{0\}$. By the induction hypothesis, $A_{\theta|_H}$ is a simple AT algebra with real rank zero. So Proposition 3.1 implies that A_θ is a simple AT algebra with real rank zero.

Now assume there is no such subgroup H_0 of rank $d-1$. Let B be as in Lemma 3.4, with

$$B^{\mathrm{t}}\theta B = \begin{pmatrix} \rho_{1,1} & \rho_{1,2} \\ (\rho_{1,2})^{\mathrm{t}} & \rho_{2,2} \end{pmatrix},$$

and where in particular all the entries of $\rho_{1,2}$ are in $\mathbb{Z}$. Then $A_{B^{\mathrm{t}}\theta B} \cong A_\rho$ for

$$\rho = \begin{pmatrix} \rho_{1,1} & 0 \\ 0 & \rho_{2,2} \end{pmatrix}.$$

Using the definitions of A_ρ, $A_{\rho_{1,1}}$, and $A_{\rho_{2,2}}$ as universal algebras on generators and relations, one easily checks that $A_\rho \cong A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$. By the induction hypothesis, both $A_{\rho_{1,1}}$ and $A_{\rho_{2,2}}$ are simple AT algebras with real rank zero. Therefore $A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$ is a simple direct limit, with no dimension growth, of homogeneous C*-algebras. Since it has a unique tracial state, Theorems 1 and 2 of [2] imply that A_ρ has stable rank one and real rank zero. This algebra has weakly unperforated K-theory by Theorem 6.1 of [32]. (Actually, this is true for any direct limit of the

type at hand.) It now follows from Theorem 4.6 of [21] that $A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$ has tracial rank zero. So Corollary 3.3 shows that A_θ has tracial rank zero. ∎

We note that one could use the earlier Theorem 3.11 of [11] to show that $A_{\rho_{1,1}} \otimes A_{\rho_{2,2}}$ is an AT algebra with real rank zero, from which it follows that this algebra has tracial rank zero. The use of H. Lin's classification theorem, Theorem 5.2 of [23], remains essential, because we can only prove that crossed products and fixed point algebras of actions by finite cyclic groups with the tracial Rokhlin property preserve tracial rank zero, not that they the property of being an AT algebra or even an AH algebra.

Remark 3.6. Since the paper [14] remains unpublished, it is worth pointing out that the proof of Theorem 3.5 does not actually depend on this paper. In the proof of Lemma 1.13, we need to know that the ordinary irrational rotation algebras have real rank zero, but this follows from Remark 6 of [10]. In the proof of Proposition 2.10, we need to know that traces determine order on projections in A_θ whenever A_θ is simple. The proof of this in [3] does not rely on [14], and in any case an independent proof (valid whenever θ is not purely rational) is contained in [32]. And in the application of Theorem 6.4 of [18] in the proof of Proposition 3.1, we use the fact that A_α has a unique tracial state, rather than real rank zero, to show that Kishimoto's conditions hold.

To finish the proof that A_θ is an AT algebra, we use the following consequence of H. Lin's classification theorem [23] for C*-algebras with tracial rank zero. This result is well known, but we have been unable to find it explicitly in the literature. Accordingly, we give it here. We include the AH and AF cases as well as the AT case for convenient reference elsewhere, because they have the same proof, although we do not use them here.

Recall that an AH algebra is a direct limit of finite direct sums of corners of homogeneous C*-algebras whose primitive ideal spaces are finite CW complexes. See, for example, the statement of Theorem 4.6 of [12], except that we omit the restrictions there on the type of CW complexes which may appear; or see 2.5 of [19].

Proposition 3.7. Let A be a simple infinite dimensional separable unital nuclear C*-algebra with tracial rank zero and which satisfies the Universal Coefficient Theorem (Theorem 1.17 of [36]). Then A is a simple AH algebra with real rank zero and no dimension growth. If $K_*(A)$ is torsion free, then A is an AT algebra. If, in addition, $K_1(A) = 0$, then A is an AF algebra.

Proof. Theorems 6.11 and 6.13 of [20] show that $K_0(A)$ is weakly unperforated and satisfies the Riesz interpolation property (equivalently, by Proposition 2.1 of [13], the Riesz decomposition property). We now apply Theorem 4.20 of [12] to find a unital AH algebra B with real rank zero and no dimension growth whose ordered scaled K-theory is the same as that of A. Since A is simple, so is the partially ordered group $K_0(A)$, and therefore B is also simple. If $K_*(A)$ is torsion free, we claim that there is a simple AT algebra B with real rank zero whose ordered scaled K-theory is the same as that of A. To prove this, note that $K_0(A)$ can't be $\mathbb{Z}$ because A has real rank zero; then we apply the proof of Theorem 8.3 of [9]. (As noted in the introduction to [9], the part of the order involving K_1 is irrelevant in the simple case.) We can certainly take the groups in the direct limit decomposition to be torsion free, so that the proof shows that all the algebras in the direct system

constructed there may be taken to have primitive ideal space the circle or a point. Then Theorem 4.3 of [9] shows they may all be taken to have primitive ideal space the circle. This gives the required AT algebra B. Finally, if in addition $K_1(A) = 0$, following [7] we may find a simple AF algebra B whose ordered scaled K-theory is the same as that of A.

Proposition 2.6 of [19] (with $\mathcal{C}$ as defined in 2.5 of [19]) implies that simple AH algebras with real rank zero and no dimension growth have tracial rank zero. In particular, B has tracial rank zero. So the classification theorem for C*-algebras with tracial rank zero, Theorem 5.2 of [23], implies that $A \cong B$. ∎

Theorem 3.8. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 2$. Then A_θ is a simple AT algebra with real rank zero.

Proof. Using Theorem 1.17 of [36] (see the preceding discussion for the definition of $\mathcal{N}$), it follows from Lemma 1.5 that A_θ satisfies the Universal Coefficient Theorem. Clearly A_θ is separable and nuclear. Since

$$K_0(A_\beta) \cong K_1(A_\beta) \cong \mathbb{Z}^{2^{n-1}}$$

by Lemma 1.5 and by repeated application of the Pimsner-Voiculescu exact sequence [31], Theorem 3.5 and Proposition 3.7 imply that A_θ is a simple AT algebra with real rank zero. ∎

We now consider the isomorphism and Morita equivalence classification of simple higher dimensional noncommutative tori. For a nondegenerate skew symmetric real $d \times d$ matrix θ, Elliott [8] has determined the range of the unique tracial state τ_θ acting on $K_0(A_\theta)$, in terms of the "exterior exponential" $\exp_\wedge(\theta)\colon \Lambda^{\mathrm{even}}\mathbb{Z}^d \to \mathbb{R}$. We regard θ as a linear map from $\mathbb{Z}^d \wedge \mathbb{Z}^d$ to $\mathbb{R}$. Following [8], if $\varphi\colon \Lambda^k\mathbb{Z}^d \to \mathbb{R}$ and $\psi\colon \Lambda^l\mathbb{Z}^d \to \mathbb{R}$ are linear, we take, by a slight abuse of notation, $\varphi \wedge \psi\colon \Lambda^{k+l}\mathbb{Z}^d \to \mathbb{R}$ to be the functional obtained from the alternating functional on $(\mathbb{Z}^d)^{k+l}$ defined as the antisymmetrization of

$$(x_1, x_2, \ldots, x_{k+l}) \mapsto \varphi(x_1 \wedge x_2 \wedge \cdots \wedge x_k)\psi(x_{k+1} \wedge x_{k+2} \wedge \cdots \wedge x_{k+l}).$$

In a similar way, we take $\varphi \oplus \psi\colon \Lambda^k\mathbb{Z}^d \oplus \Lambda^l\mathbb{Z}^d \to \mathbb{R}$ to be $(\xi, \eta) \mapsto \varphi(\xi) + \psi(\eta)$. Then by definition

$$\exp_\wedge(\theta) = 1 \oplus \theta \oplus \tfrac{1}{2}(\theta \wedge \theta) \oplus \tfrac{1}{6}(\theta \wedge \theta \wedge \theta) \oplus \cdots \colon \Lambda^{\mathrm{even}}\mathbb{Z}^d \to \mathbb{R}.$$

Elliott's result for the nondegenerate case is then as follows.

Theorem 3.9. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix. Then there is an isomorphism $h\colon K_0(A_{\theta_j}) \to \Lambda^{\mathrm{even}}\mathbb{Z}^d$ such that $\exp_\wedge(\theta) \circ h = (\tau_{\theta_j})_*$, and such that $h([1])$ is the standard generator $1 \in \Lambda^0(\mathbb{Z}^d) = \mathbb{Z}$.

Proof. This is 1.3, Theorem 2.2, and Theorem 3.1 of [8]. ∎

For the Morita equivalence result, we need the following lemma.

Lemma 3.10. Let G_1 and G_2 be finitely generated free abelian groups with the same rank, and let $f_1\colon G_1 \to \mathbb{R}$ and $f_2\colon G_2 \to \mathbb{R}$ be homomorphisms with the same range. Then there exists an isomorphism $g\colon G_1 \to G_2$ such that $f_2 \circ g = f_1$.

Proof. Let $D \subset \mathbb{R}$ be the common range. Then D is a finitely generated subgroup of $\mathbb{R}$, so is free. Let $t_1, t_2, \ldots, t_k \in D$ form a basis.

Choose $\eta_1, \eta_2, \ldots, \eta_k \in G_1$ such that $f_1(\eta_j) = t_j$ for $1 \leq j \leq k$. Choose elements $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_n \in G_1$ which form a basis for $\mathrm{Ker}(f_1)$. We claim that $\eta_1, \eta_2, \ldots, \eta_n$ form a basis for G_1. To prove linear independence, suppose $\sum_{j=1}^{n} \alpha_j \eta_j = 0$. Apply f_1 to get $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$, and use linear independence of $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_n$. To show that they span G_1, let $\eta \in G_1$, choose $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{Z}$ such that $f_1(\eta) = \sum_{j=1}^{k} \alpha_j t_j$, and use $\eta - \sum_{j=1}^{k} \alpha_j \eta_j \in \mathrm{Ker}(f_1)$ to write this element as an integer combination of $\eta_{k+1}, \eta_{k+2}, \ldots, \eta_n$.

Similarly, there is a basis $\mu_1, \mu_2, \ldots, \mu_n$ for G_2 such that $g(\mu_j) = t_j$ for $1 \leq j \leq k$ and $g(\mu_j) = 0$ for $k + 1 \leq j \leq n$. (It has the same number of elements because G_1 and G_2 have the same rank.) The required isomorphism g is now defined by specifying $g(\eta_j) = \mu_j$ for $1 \leq j \leq n$. ∎

Theorem 3.11. Let θ_1 and θ_2 be nondegenerate skew symmetric real $d \times d$ matrices, with $d \geq 2$. Then A_{θ_1} is strongly Morita equivalent to A_{θ_2} if and only if there exists $\lambda > 0$ such that $\exp_\wedge(\theta_1)$ and $\lambda \exp_\wedge(\theta_2)$ have the same range.

Proof. By Theorem 3.9, the condition is equivalent to the existence of $\lambda > 0$ such that $(\tau_{\theta_1})_*$ and $\lambda(\tau_{\theta_2})_*$ have the same range.

The condition is certainly necessary. For sufficiency, use Lemma 3.10 to find an isomorphism $g \colon K_0(A_{\theta_1}) \to K_0(A_{\theta_2})$ such that $\lambda(\tau_{\theta_2})_* \circ g = (\tau_{\theta_1})_*$. Since A_{θ_2} has tracial rank zero, there are $n \in \mathbb{N}$ and a projection $p \in M_n(A_{\theta_2})$ such that $[p] = g([1])$. Then A_{θ_1} and $pM_n(A_{\theta_2})p$ have isomorphic Elliott invariants, so $A_{\theta_1} \cong pM_n(A_{\theta_2})p$ by Theorem 5.2 of [23]. ∎

For the isomorphism classification, we have:

Theorem 3.12. Let θ_1 and θ_2 be nondegenerate skew symmetric real $d \times d$ matrices, with $d \geq 2$. Then $A_{\theta_1} \cong A_{\theta_2}$ if and only if there is an isomorphism $g \colon \Lambda^{\mathrm{even}}\mathbb{Z}^d \to \Lambda^{\mathrm{even}}\mathbb{Z}^d$ such that $\exp_\wedge(\theta_2) \circ g = \exp_\wedge(\theta_1)$ and such that g sends the standard generator $1 \in \Lambda^0(\mathbb{Z}^d) = \mathbb{Z}$ to itself.

Proof. By Theorem 3.9, the condition is equivalent to isomorphism of the Elliott invariants of A_{θ_1} and A_{θ_2}. Apply Theorem 5.2 of [23]. ∎

One might hope that it would suffice to require that $(\tau_{\theta_1})_*$ and $(\tau_{\theta_2})_*$ have the same range. We show by example that this is not true.

Example 3.13. Choose $\beta, \gamma \in \mathbb{R}$ such that $1, \beta, \gamma$ are linearly independent over $\mathbb{Q}$. Set

$$\theta_1 = \begin{pmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \frac{2}{5} \\ -\gamma & -\frac{2}{5} & 0 \end{pmatrix} \quad \text{and} \quad \theta_2 = \begin{pmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \frac{1}{5} \\ -\gamma & -\frac{1}{5} & 0 \end{pmatrix}.$$

We claim that:

(1) θ_1 and θ_2 are nondegenerate.
(2) $(\tau_{\theta_1})_*$ and $(\tau_{\theta_2})_*$ have the same range.
(3) $A_{\theta_1} \not\cong A_{\theta_2}$.

Set $\lambda_1 = \frac{2}{5}$ and $\lambda_2 = \frac{1}{5}$, giving

$$\theta_l = \begin{pmatrix} 0 & \beta & \gamma \\ -\beta & 0 & \lambda_l \\ -\gamma & -\lambda_l & 0 \end{pmatrix}.$$

To prove (1), we verify the condition of Lemma 1.7. Thus, suppose $x \in \mathbb{Q}^d$ and $\langle x, \theta_l y \rangle \in \mathbb{Q}$ for all $y \in \mathbb{Q}^d$. Putting $y = (1, 0, 0)$, we get $-\beta x_2 - \gamma x_3 \in \mathbb{Q}$, whence $x_2 = x_3 = 0$. Putting $y = (0, 1, 0)$ and using $x_3 = 0$, we get $\beta x_1 \in \mathbb{Q}$, whence $x_1 = 0$. So $x = 0$, proving nondegeneracy.

For the proofs of (2) and (3), apply Theorem 3.9 to θ_1 and θ_2, obtaining isomorphisms h_1 and h_2. Use $\exp_{\wedge}(\theta_l) \circ h_l = (\tau_{\theta_l})_*$, and identify $\Lambda^2 \mathbb{Z}^3$ with $\mathbb{Z}^3$ in such a way that θ_l, regarded as a linear map from $\mathbb{Z}^d \wedge \mathbb{Z}^d$ to $\mathbb{R}$, sends the standard basis elements to λ_l, β, and γ. Thus there are isomorphisms $h_l \colon K_0(A_{\theta_l}) \to \mathbb{Z}^4$ such that $h_l([1]) = (1, 0, 0, 0)$ and such that the map $f_l \colon \mathbb{Z}^4 \to \mathbb{R}$ given by $n \mapsto n_1 + \lambda_l n_2 + \beta n_3 + \gamma n_4$ satisfies $(\tau_{\theta_l})_* = f_l \circ h_l$.

We prove (2) by showing that the ranges of f_1 and f_2 are equal to $\frac{1}{5}\mathbb{Z} + \beta\mathbb{Z} + \gamma\mathbb{Z}$. This is obvious for f_2. Also, it is obvious that

$$1, \tfrac{2}{5}, \beta, \gamma \in f_1(\mathbb{Z}^4) \subset \tfrac{1}{5}\mathbb{Z} + \beta\mathbb{Z} + \gamma\mathbb{Z},$$

whence also $f_1(\mathbb{Z}^4) = \frac{1}{5}\mathbb{Z} + \beta\mathbb{Z} + \gamma\mathbb{Z}$.

We turn to the proof of (3). It suffices to show that there is no isomorphism $g \colon \mathbb{Z}^4 \to \mathbb{Z}^4$ such that $f_2 \circ g = f_1$ and $g(1, 0, 0, 0) = (1, 0, 0, 0)$. Suppose we had such a map g. The equation $g(1, 0, 0, 0) = (1, 0, 0, 0)$ determines the first column of the matrix of g. Since

$$f_2(0, 2, 0, 0) = \tfrac{2}{5}, \quad f_2(0, 0, 1, 0) = \beta, \quad \text{and} \quad f_2(0, 0, 0, 1) = \gamma,$$

the other columns are determined by

$$g(0, 1, 0, 0) \in (0, 2, 0, 0) + \mathrm{Ker}(f_2), \quad g(0, 0, 1, 0) \in (0, 0, 1, 0) + \mathrm{Ker}(f_2),$$

and

$$g(0, 0, 0, 1) \in (0, 0, 0, 1) + \mathrm{Ker}(f_2).$$

Now

$$\mathrm{Ker}(f_2) = \{(-r, \, 5r, \, 0, \, 0) \in \mathbb{Z}^4 \colon r \in \mathbb{Z}\}.$$

So there are $r, s, t \in \mathbb{Z}$ such that

$$g = \begin{pmatrix} 1 & -r & -s & -t \\ 0 & 2 + 5r & 5s & 5t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now g can be invertible only if $2 + 5r = \pm 1$, which is not possible for $r \in \mathbb{Z}$. This proves (3).

We do, however, get the following result. Recall that the opposite algebra A^{op} of a C*-algebra A is the algebra A with the multiplication reversed but all other operations, including the scalar multiplication, the same.

Corollary 3.14. Let θ be a nondegenerate skew symmetric real $d \times d$ matrix, with $d \geq 2$. Then $(A_\theta)^{\mathrm{op}} \cong A_\theta$.

Proof. Using classification (for example, Theorem 5.2 of [23]), one sees that every simple AT algebra A with real rank zero is isomorphic to its opposite algebra, because the ordered K-theory of A^{op} is the same as the ordered K-theory of A. ∎

As far as we know, it is unknown whether $(A_\theta)^{\mathrm{op}} \cong A_\theta$ for general degenerate θ.

Corollary 3.15. Under the hypotheses of Proposition 2.10, and with the additional condition $n \neq 1$, the action of $\mathbb{Z}_n$ on A_θ generated by α does not have the Rokhlin property.

Proof. Clearly this action is trivial on $K_*(A_\theta)$. It follows from Theorem 3.8 that the hypotheses on the algebra in Theorem 3.5 of [16] are satisfied. This theorem implies, in particular, that if the action had the Rokhlin property, then A_θ would be isomorphic to its tensor product with the n^∞ UHF algebra. The K-theory shows this is impossible. ∎

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